ON THE MULTIPLICATIVE REPRESENTATION OF INTEGERS

BY

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Dedicated to my friend A. D. Wallace on the occasion of his 60th birthday.

ABSTRACT

Let $a_1 < a_2 < \cdots$ be an infinite sequence of integers. Denote by $g(n)$ the number of solutions of $n = a_i \cdots a_j$. If $g(n) > 0$ for a sequence n of positive upper density then lim sup $g(n) = \infty$.

Let $a_1 < a_2 < \cdots$ be an infinite sequence of integers and denote by $f(n)$ the number of solutions of $n = a_i + a_j$. An old conjecture of Turán and myself states that if $f(n) > 0$ for all $n > n_0$ then $\limsup_{n=\infty} f(n) = \infty$. A stronger conjecture (which nevertheless might be easier to attack) states that if $a_k < ck^2$ then $\limsup_{n=\infty} f(n) = \infty$. Both these conjectures seem rather deep. I could only prove that $a_k < ck^2$ implies that the sums $a_i + a_j$ can not all be different [6] ($c, c_1, c_2, \cdots$ denote absolute constants).

In view of the difficulty of these conjectures it is perhaps surprising that the multiplicative analogues of these conjectures though definitely non-trivial are not too hard to settle. In fact I shall prove the following.

THEOREM 1. *Let $b_1 < b_2 < \cdots$ be an infinite sequence of integers. Denote by $g(n)$ the number of solutions of $n = b_i b_j$. Then*

(1) $$g(n) > 0 \text{ for all } n > n_0$$

implies

(2) $$\limsup_{n=\infty} g(n) = \infty$$

Define

$$B(x) = \sum_{b_i \leq x} 1$$

A well known theorem of Raikov [5] states that (1) implies that for infinitely many x

(3) $$B(x) > c_1 x/(\log x)^{1/2}$$

Thus to prove Theorem 1 it will suffice to show that if (3) holds for infinitely many x then (2) follows. In fact I shall prove stronger results.

Denote by $u_t(n)$ the smallest integer so that if $b_1 < \cdots < b_t \leqq n$, $t = u_t(n)$ is any

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sequence of integers then for some m, $g(m) \geqq l$. Theorem 1 would follow from $u_l(n) = 0(n/(\log n)^{1/2})$.

THEOREM 2.

$$u_{2^k}(n) < c_2 \frac{n}{\log n}(\log\log n)^{k+1}$$

In a previous paper I [1] proved that

$$\Pi(n) + c_3 n^{3/4}/(\log n)^{3/2} < u_2(n) < \Pi(n) + c_4 n^{3/4}. \tag{4}$$

$\Pi(n)$ denotes the number of primes not exceeding n and $\Pi_k(n)$ denotes the number of integers $m \geqq n$ the number of distinct prime factors of which does not exceed k. The right side of (4) can in fact be stregthened to

$$u_2(n) < \Pi(n) + c_5 n^{3/4}/(\log n)^{3/2} \tag{5}$$

I do not prove (5) in this paper.

(4) and (5) suggest the possibility of obtaining an asymptotic formula with an error term for $u_l(n)$ also for $l > 2$. I am going to outline the proof of

THEOREM 3. *Let* $2^{k-1} < l \leqq 2^k$. Then

$$u_l(n) = (1 + o(1) \frac{n(\log\log n)^{k-1}}{(k-1)!\log n}.$$

Finally I am going to prove the following

THEOREM 4. *To every c and l there is an $n_0 = n_0(c, l)$ so that if $n > n_0$ and $b_1 < \cdots < b_s \leqq n$ is such that the number $N(n)$ of integers $t < n$ which can be written in the form $b_i b_j$ is greater than cn then there is an m with $g(m) > l$.*

Theorem 4 clearly implies Theorem 1, but not Theorems 2 and 3.

Our main tool will be the following

LEMMA. *Let $S_1, \cdots, S_r$ be r sets of integers, S_i has N_i elements ($N_1 > \cdots > N_r$) $x_j^{(i)}$, $1 \leqq j \leqq N_r$. Let $u_1 < u_2 < \cdots < u_t$ be a sequence of integers where each u_j, $1 \leqq j \leqq t$ is of the form $\prod_{i=1}^{r} x_j^{(i)}$(i.e. every u can be written as the product of r integers one from each S_i). Then if*

$$t > \frac{3^r r^r}{N_r^1 2^{r-1}} \prod_{i=1}^{r} N_i \tag{6}$$

there is an m so that the number of solutions of

$$m = u_{j_1} u_{j_2}$$

is at least 2^{r-1}.

To each integer of S_i, $1 \leqq i \leqq r$ we make correspond a vertex and to each

$u_j = \prod_{i=1}^{r} x_{j_i}^{(i)}$ we make correspond the r-tuple $\{x_{ji}\}^{(i)}_{,}$ $1 \leqq i \leqq r$. Thus we obtain an r-Graph [2] $G^{(r)}(\sum_{i=1}^{r} N_i\,;\,t)$ and if t satisfies (6) then by the corollary of Theorem 1 of [2] there are integers $x_1^{(i)}, x_2^{(i)}$ in S_i, $1 \leqq i \leqq r$ so that all the 2^r integers

$$\prod_{i=1}^{r} x_\lambda^{(i)}, \quad \lambda = 1 \text{ or } 2$$

are u's. Thus $\prod_{i=1}^{r}(x_1^{(i)} x_2^{(i)} = u_{j_1} u_{j_2}$ has at least 2^{r-1} solutions, which completes the proof of the Lemma.

Let now $b_1 < \cdots < b_s \leqq n$ be a sequence of integers for which $g(m) < 2^k$ for all m. To prove Theorem 2 we have to show

$$s < \frac{c_2 n (\log\log n)^{k+1}}{\log n}. \tag{7}$$

To prove (7) we split the b's into two classes.

In the first class are the b's which can not be written in the form ($\exp z = e^z$)

$$\prod_{i=1}^{k+1} e_i,\; e_i > \exp((\log\log n)^2). \tag{8}$$

Denote these b's by $b_1', \cdots, b_{s_1}'$ and write $b_i = u_i v_i'$ where all prime factors of u_i are not exceeding $\exp((\log\log n)^2)$, and all prime factors of v_i are greater than $\exp((\log\log n)^2)$. By (8) v_i has at most k prime factors (for otherwise v_i and therefore $b_i' = u_i v_i$ would be of the form (8). Further a simple argument shows that $u_i < \exp((2k+2)(\log\log n)^2)$ (for otherwise u_i and therefore b_i' would be of the form (8)). But then clearly

$$s_1 \leqq \sum_i \Pi_k\left(\frac{n}{u_i}\right) \leqq \sum{}' \Pi_k\left(\frac{n}{t}\right) \tag{9}$$

where the dash in the summation indicates that $1 \leqq t < \exp((2k+2)(\log\log n)^2)$. Now by a theorem of Landau [4]

$$\Pi_k(x) = (1 + o(1)) \frac{x (\log\log x)^{k-1}}{(k-1)!\log x}. \tag{10}$$

Thus from (9) and (10) we obtain by a simple computation

$$s_1 < c_6 n (\log\log n)^{k+1} / \log n. \tag{11}$$

Denote now by $b''_1, \cdots b''_{s_2}$ the b's of the form (8). If $c_2 > c_6$ and (7) would be false, we would have from (11)

$$s_2 > (c_2 - c_6) \frac{n (\log\log n)^{k+1}}{\log n}. \tag{12}$$

Put for $1 \leqq j \leqq s_2$ (we write $e_j^{(i)}$ instead of $e_{j_i}^{(i)}$)

$$(13)\qquad \begin{cases} b_j'' = \prod_{i=1}^{k+1} e_j^{(i)},\ 2^{\lambda_j^{(i)}} \leqq e_j^{(i)} < 2^{1+\lambda_j^{(i)}} \\ (\log\log n)^2 \leqq \lambda_j^{(i)} \leqq \dfrac{\log n}{\log 2}. \end{cases}$$

To each b_j'' we make correspond the $(k+1)$-tuple

$$(14)\qquad \{\lambda_j^{(i)}\}, \qquad 1 \leqq i \leqq k+1.$$

By (13) the number of possible choices of the $(k+1)$-tuples (14) is for $n > n_0$ less than $(\log n/\log 2)^{k+1}$. Thus by (12) there is a $(k+1)$-tuple $(\lambda_1, \cdots, \lambda_{k+1})$ which corresponds to more than $n/(\log n)^{k+3} b'''$'s say $b_1''', \cdots, b_{s_3}'''$

$$(15)\qquad s_3 > n/(\log n)^{k+3}$$

Now we apply our Lemma with $r = k+1$. The sets S_i are the integers in $(2^{\lambda_i}, 2^{1+\lambda_i})$, thus $N_i = 2^{\lambda_i}$, and by $b_i''' \leqq n$ we have

$$\prod_{i=1}^{k+1} N_i \leqq 2^{\sum_{i=1}^{k+1} \lambda_1} \leqq n$$

By (15) and $\lambda_i \geqq (\log\log n)^2$ a simple computation shows that $s_3 = t$ clearly satisfies (6). Thus by our Lemma there is an m for which $m = b_{l_1}''' b_{l_2}'''$ has at least 2^k solutions, which proves Theorems 1 and 2.

COROLLARY. *Let $b_1 < \cdots$ be an infinite sequence of integers so that every $n > n_0$ can be written as the product of k or fewer b's. Then* $\limsup_{n=\infty} g(n) = \infty$.

Raikov's theorem implies that for infinitely many x $B(x) > cx/(\log x)^{1/k}$. Thus the corollary follows from Theorem 2.

Now we prove Theorem 4. We shall show that there is an $\varepsilon = \varepsilon(c) > 0$ so that to every T there is an $n_0 = n_0(T, \varepsilon)$ for which, $n > n_0$, $N(n) > cn$ implies that there is an $L > T$ satisfying.

$$(16)\qquad B(L) > \varepsilon L/(\log L)^{1/2}.$$

(16) by Theorem 2 implies Theorem 4.

(16) implies Raikov's theorem with $\varepsilon = c_3$. Our proof of (16) will not use Raikov's theorem but we will use his method.

We evidently have

$$(17)\qquad cn < N(n) \leqq \sum_i B\left(\frac{n}{b_i}\right) = \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3$$

where in $\sum_1$ $b_i \leqq T$, in $\sum_2$ $T < b_i < n/T$ and in $\sum_3$ $b_i \geqq n/T$. Clearly

(18) $$\sum_1 \leqq TB(n)$$

and

(19) $$\sum_3 \leqq TB(n).$$

(19) follows from the fact that there are most $B(n)$ summands in $\sum_3$ and each summand is $\leqq T$. If (16) does not hold then for every $L > T$

(20) $$B(L) \leqq \varepsilon L/(\log L)^{1/2}.$$

Thus from (18), (19) and (20) we have for $n > n_0\ (T, \varepsilon)$

(21) $$\sum_1 + \sum_3 \leqq 2TB(n) \leqq 2T\varepsilon n/(\log n)^{1/2} < cn/2.$$

From (20) we further have

(22) $$\sum_2 \leqq \sum_{T<b_i<n/T} \varepsilon n/b_i \left(\log \frac{n}{b_i}\right)^{1/2} .$$

Now from (20) we have by a simple argument that for $b_i > T$, $b_i > i(\log i)^{1/2}$. Thus from (22) we obtain by a simple computation

(23) $$\sum_2 < \varepsilon n \sum_{i=2}^{n} \frac{1}{i(\log i)^{1/2}\left(\log \frac{n}{i}\right)^{1/2}} < c_7 \varepsilon n < \frac{c}{2} n$$

if ε is sufficiently small. (21) and (23) contradicts (17), thus (20) can not hold for all $L > T$ (or (16) holds for some $L > T$) which completes the proof of Theorem 4.

The following problem can now be put: Assume that (1) holds. What can be said about

$$F(n) = \max_{m \leqq n} g(m) .$$

I can prove that there are two constants α_1 and α_2 so that (1) implies for $n > n_0$

(24) $$F(n) > (\log n)^{\alpha_1}.$$

But there exists a sequence $b_1 < \cdots$ for which (1) is satisfied and for all n

(25) $$F(n) < (\log n)^{\alpha_2}.$$

In this paper I do not give the proof of (24) and (25) but only remark that the proof of (24) is a refinement of the proof of Theorem 2 and the proof of (25) uses probabilistic arguments similar to the ones used in [3].

Now we outline the proof of Theorem 3. By (10) Theorem 3 implies that for $2^{k-1} < l \leqq 2^k$ $u_l(n) = (1 + o(1))\Pi_k(n)$.

First we show

(26) $$u_{2^{k-1}+1}(n) \geqq (1+o(1))\, \Pi_{k-1}(n) = (1+o(1))\, \frac{n(\log\log n)^{k-1}}{(k-1)!}.$$

Denote by $v_1^{(k)} < \cdots < v_{t_k}^{(k)} \leqq n$ the set of integers of the form

$$\frac{n}{\log n} < \prod_{i=1}^{k} p_i < n, \qquad p_{i+1} < p_i^{1/k^2}, \ p_k > (\log n)^2. \tag{27}$$

It is a simple exercise in analytic number theory to prove by induction with repect to k that

$$t_k = (1 + o(1)) \frac{n(\log\log n)^{k-1}}{(k-1)!\log n}. \tag{28}$$

We leave the proof of (28) to the reader. To prove (26) we now show that for every m the number of solutions of

$$v_{j_1}^{(k)} v_{j_2}^{(k)} = m \tag{29}$$

is at most 2^{k-1}. Observe that if (29) is solvable we must have

$$m = \prod_{i=1}^{k} p_i q_i \tag{30}$$

where $\prod_{i=1}^k p_i$ and $\prod_{i=1}^k q_i$ both satisfy (27). Every solution of (29) must be of the form

$$v_{j_1}^{(k)} = \prod_{i=1}^{k} x_i^{(1)}, \quad v_{j_2}^{(k)} = \prod_{i=1}^{k} x_i^{(2)}, \quad x_1^{(1)} > \cdots > x_k^{(1)}; \quad x_1^{(2)} > \cdots > x_k^{(2)} \tag{31}$$

where the $x_i^{(1)}$ and $x_i^{(2)}$ are the p's and q's and $\prod_{i=1}^k x_i^{(1)}$ and $\prod_{i=1}^k x_i^{(2)}$ satisfy (27).

$x_i^{(1)}$ and $x_i^{(2)}$ we will call the i-th coordinate of $v_{j_1}^{(k)}$ respectively $v_{j_2}^{(k)}$. Clearly p_1 and q_1 must be the first coordinates of any possible solution of (29). To see this observe that (27) implies

$$\prod_{i=2}^{k} p_i q_i < n^{1/2} < n/\log n$$

and hence (27) can be satisfied only if the first coordinates are p_1 and q_1. Assume that the first $i-1$ coordinates of a solution $v_{j_1}^{(k)}$ of (29) has already been chosen. I claim that there are only two possible choices for the i-th coordinate of $v_{j_1}^{(k)}$. To show this it will suffice to prove that only one p and only one q can possibly occur as the i-th coordinate of $v_{j_1}^{(k)}$. If this is not so we assume that both $v_j' = x_1 \cdots x_{i-1} p_u x_{i+1} \cdots x_k$ and $v_j'' = x_1 \cdots x_{i-1} p_v x_{i+1}' \cdots x_k'$ would be solution of (29). But then clearly

$$v_j' \geqq x_1 \cdots x_{i-1} p_u, \quad v_j'' < x_1 \cdots x_{i-1} p_v^k. \tag{32}$$

Hence by (27) and (32)

$$\log n > v_j'/v_j'' > p_u/p_v^k \geqq p_u^{1/2} > \log n$$

an evident contradiction.

The fact that the first coordinates of every solution of (29) must be p_1 and q_1 and the fact that for $i > 1$ there are most two choices for the i-th coordinate of $v_{j_1}^{(k)}$ immediately implies that (29) has at most 2^{k-1} solutions. Thus by (28), (26) is proved.

To complete the proof of Theorem 3 we have to show

$$u_{2^k}(n) \leqq (1 + o(1)) \frac{n(\log\log n)^{k-1}}{(k-1)!\log n}. \tag{33}$$

To prove (33) it suffices to show that to every $\varepsilon > 0$ there is an $n_0 = n_0(\varepsilon, k)$ so that if

$$b_1 < \cdots < b_l, \ l > (1+\varepsilon) \frac{n(\log\log n)^{k-1}}{(k-1)!\log n} \tag{34}$$

is any sequence of integers then there is an m with $g(m) \geqq 2^k$. We will only outline the fairly complicated proof.

Assume that there is a sequence satisfying (34) for which $g(m) < 2^k$ for all m. We shall show that this assumption leads to a contradiction. We split the b's into five classes. In the first class are the b's which can be written in the form

$$\prod_{i=1}^{k+1} e_i, \quad e_i > (\log n)^{c_k}, \quad 1 \leqq i \leqq k+1 \tag{35}$$

where c_k is a sufficiently large absolute constant. Using (35) and our Lemma in the same way as we used (8) and our Lemma in the proof of Theorem 2 we obtain that $g(m) < 2^k$ for all m implies that the number of integers of the first class is $O((n/(\log n)^2))$. The integers of the second class have at most $k-2$ prime factors $> (\log n)^{c_k}$ and they can not be written in the form (35). In the asme way as we proved (11) we can show that the number of integers of the second class is less than $(cn(\log\log n)^{k-2})/\log n)$. The integers which do not belong to the first two classes can be written in the form

$$t \prod_{i=1}^{k-1} p_i, \ p_i > (\log n)^{c_k}, \ 1 \gneqq i \leqq k-1 \tag{36}$$

and where t can not be written as the product of two integers $> (\log n)^{c_k}$ (for otherwise our number would be of the first class). In the third class are the integers where all prime factors of t are less than $(\log n)^{\eta_1}$ where $\eta_1 = \eta_1(\varepsilon)$ is sufficiently small. We can assume $t < (\log n)^{4c_k}$ for otherwise t would be the product of two integers $> (\log n)^{c_k}$. Thus the number of integers of the third class is at most $\sum' \Pi_{k-1}((n/t))$ where the dash indicates that $t < (\log n)^{4c_k}$ and all prime factors of t are less than $(\log n)^{\eta_1}$. By a simple computation we have from (10) and $\eta_1 = \eta_1(\varepsilon)$

$$(37)\quad \sum{}' \Pi_{k-1}\left(\frac{n}{t}\right) = (1+o(1)\ \frac{n(\log\log n)^{k-2}}{(k-2)!\log n}\sum{}'\frac{1}{t} < \frac{\varepsilon}{10}\ \frac{n(\log\log n)^{k-1}}{(k-1)!\log n}$$

Thus by (37) the number of b's which belong to the first three classes is less than $(\varepsilon/2)(n(\log\log n)^{k-1})/(k-1)!\log n)$ and hence by (34) there are at least

$$(38)\qquad (1+\frac{\varepsilon}{2})\ \frac{n(\log\log n)^{k-1}}{(k-1)!\log n}$$

b's which do not belong to the first three classes. These b's can by (36) all be written in the form (p_k is the greatest prime factor of t)

$$(39)\qquad t'\prod_{i=1}^{k} p_i,\quad p_i > (\log n)^{c_k},\ \ 1 \leqq i \leqq k-1,\ \ p_k > (\log n)^{\eta_1},\ t' = \frac{t}{p_k}.$$

In the fourth class are the b's for which

$$(40)\qquad t' < (\log n)^{\eta_2} \text{ where } \eta_2 = \eta_2(\eta_1)$$

is sufficiently small. We shall now show that our assumption $g(m) < 2^k$ for all m implies that the number N of integers of the fourth class is less than

$$(41)\qquad N < \left(1+\frac{\varepsilon}{4}\right)\frac{n(\log\log n)^{k-1}}{(k-1)!\log n}.$$

If C is any set of integers $N(C)$ will denote the number of integers of this class.

Let b_i be any integer of the fourth class, b_i can be written (uniquely) in the form (39) and by I_t, we denote the set of integers b_i/t'. The integers in $I_{t'}$ have all k prime factors. If (41) does not hold then (in $\sum{}'\ t' < (\log n)^{\eta_2}$)

$$(42)\qquad N = \sum_{t'}{}' N(I_{t'}) \geqq \left(1+\frac{\varepsilon}{4}\right)\frac{n(\log\log n)^{k-1}}{(k-1)!\log n}.$$

We evidently have ($I_{t'} \cap I_{t''}$ is the set of integers belonging to both $I_{t'}$ and $I_{t''}$ In $\sum{}''\ t' \neq t''$)

$$(43)\qquad \begin{cases} \sum_{t'}{}' N(I_{t'}) \leqq \Pi_k(n) + \sum_{t',t''}{}'' N(I_{t'} \cap I_{t''}) < \\ \Pi_k(n) + (\log n)^{2\eta_2} \max\limits_{t',t''} \ N(I_{t'} \cap I_{t''}). \end{cases}$$

From (42), (43) and (10) we obtain that for $n > n_0$

$$(44)\qquad \max_{t',t''} N(I_{t'} \cap I_{t''}) > \frac{\varepsilon}{10}\ \frac{n(\log\log n)^{k-1}}{(k-1)!(\log n)^{1+2\eta_2}} > \frac{n}{(\log n)^{1+3\eta_2}}$$

Hence there are values of t' and t'' say $t^{(1)}$ and $t^{(2)}(t^{(1)} \neq t^{(2)})$ for which (44) holds. We are going to prove that (44) implies that there are primes $p_i^{(1)}$, $p_i^{(2)}$, $1 \leqq i \leqq k$ so that all the 2^k products

$$(45) \qquad \prod_{i=1}^{k} p_i^{(\lambda)}, \quad \lambda = 1 \text{ or } 2$$

belong to $I_{t_1} \cap I_{t_2}$. But then all the 2^{k+1} integers.

$$t^{(\lambda)} \prod_{i=1}^{k} p_i^{(\lambda)}, \quad \lambda = 1 \text{ or } 2$$

are b's of the fourth class. Thus for $m = t^{(1)}t^{(2)}\prod_{i=1}^{k} p_i^{(1)}p_i^{(2)}$ we have $g(m) \geqq 2^k$, which contradicts our assumption, hence (41) is proved.

Thus we only have to show that if the primes $p_i^{(\lambda)}$, $1 \leqq i \leqq k$ with the property (45) do not exist then (44) can not hold. This will be accomplished by arguments similar to but more complicated than the ones used in the proof of Theorem 2.

We will only outline the argument. Denote by

$$(46) \qquad r_1 < \cdots < r_l, \quad l > n/(\log n)^{1+3\eta_2}$$

the integers belonging to $I_{t_1} \cap I_{t_2}$. By (39) each r_i is the product of k primes each greater than $(\log n)^{\eta_1}$. As in the proof of Theorem 2 we make correspond to $r_j = \prod_{i=1}^{k} p_i$ the k-tuple

$$(47) \qquad (\lambda_1, \cdots, \lambda_k), \quad \lambda_1 > \cdots > \lambda_k, \quad 2^{\lambda_i} < p_i < 2^{1+\lambda_i}$$

Denote by $N(\lambda_1, \cdots, \lambda_k)$ the number of r's corresponding to the k-tuple $(\lambda_1, \cdots, \lambda_k)$. We shall show that

$$(48) \qquad N(\lambda_1, \cdots, \lambda_k) \leqq 2^{\sum_{i=1}^{k}\lambda_i} \left(\prod_{i=1}^{k} \lambda_i \right)^{-1} 2^{-\lambda_k/2^{k+1}}.$$

By the prime number theorem the number of primes p_i satisfying (47) is $(1 + o(1))(2^{\lambda_i}/\lambda_i \log 2)$. Now we apply our Lemma with $r = k$ and

$$(49) \qquad N_r = (1 + o(1)) \frac{2^{\lambda_k}}{\lambda_k \log 2} > 2^{\lambda_k/2}, \quad 2^{\lambda_k} > \frac{1}{2}(\log n)^{\eta_1}.$$

We obtain by the Lemma by a simple argument that if

$$(50) \quad N(\lambda_1, \cdots, \lambda_k) > (1 + o(1)) 2^{\sum_{i=1}^{k}\lambda_i} \left(\prod_{i=1}^{k} \lambda_i \right)^{-1} (\log 2)^{-k}\, 2^{-\lambda_k/2^k} > 2^{\sum_{i=1}^{k}\lambda_i} \left(\prod_{i=1}^{k} \lambda_i \right)^{-1} 2^{-\lambda_k/2^{k+1}}$$

then primes $p_i^{(1)}$; $p_i^{(2)}$; $1 \leqq i \leqq k$ exist so that the numbers (45) are all r_i's and we have assumed that such primes do not exist. Thus (50) is false or (48) is proved. (48) clearly implies (the dash indicates that $2^{\sum_{i=1}^{k}\lambda_i} \leqq n$ and $2^{\lambda_k} > \frac{1}{2}(\log n)^{\eta_1}$

(51) $$l = \sum' N(\lambda_1, \cdots, \lambda_k) < \sum' 2^{\sum_{i=1}^{k} \lambda_i} \left(\prod_{i=1}^{k} \lambda_i \right)^{-1} 2^{-\lambda_k/2^{k+1}}$$

By an elementary but somewhat lengthy argument (using elementary inequalities) which we supress we obtain that (51) implies

(52) $$l < n/(\log n)^{1+3\eta_2}$$

if $\eta_2 = \eta_2(\eta_1)$ is sufficiently small. (52) contradicts (46) and this contradiction proves that (44) can not hold, which finally proves (41).

The remaining b's are in the fifth class. By (41) these integers can be written in the form

(53) $$t' \prod_{i=1}^{k} p_i, p_i > (\log n)^{c_k}, 1 \leqq i \leqq k-1, p_k > (\log n)^{\eta_1}, (\log n)^{\eta_2} < t' < (\log n)^{4c_k}$$

(if $t' \geqq (\log n)^{4c_k}$ then a simple argument would show that our $t' \prod_{i=1}^{k} p_i$ can be written in the form (35) and hence belongs to the first class). By (38) and (41) there are at least

(54) $$\frac{\varepsilon}{4} \frac{n(\log\log n)^{k-1}}{(k-1)!\log n} > \frac{n}{\log n}$$

b's of the fifth class. To each such b we make correspond a $(k+1)$-tuple $(\lambda_1, \cdots, \lambda_{k+1})$, $\lambda_1 > \cdots > \lambda_{k+1}$ satisfying

(55) $$2^{\lambda_i} < p_i < 2^{1+\lambda_i}, \ 1 \leqq i \leqq k, \quad 2^{\lambda_{k+1}} \leqq t < 2^{1+\lambda_{k+1}}$$
$$2^{\lambda_i} > \tfrac{1}{2}(\log n)^{\eta_1}, \quad 1 \leqq i \leqq k \ , \ \tfrac{1}{2}(\log n)^{\eta_2} < 2^{\lambda^{k+1}} \leqq (\log n)^{4c_k},$$
$$2^{\sum_{i=1}^{k+1} \lambda_i} \leqq n.$$

Denote by N_1 $(\lambda_1, \cdots, \lambda_{k+1})$ the number of b's of the fifth class belonging to to $(\lambda_1, \cdots, \lambda_{k+1})$. By (54) we have

(56) $$\sum' N_1(\lambda_1, \cdots, \lambda_{k+1}) > \frac{n}{\log n}$$

where the dash indicates that

(57) $$2^{\sum_{i=1}^{k+1} \lambda_i} \leqq n, \ 2^{\lambda_i} > \tfrac{1}{2}(\log n)^{\eta_1}, \ 1 \leqq i \leqq k, \ \tfrac{1}{2}(\log n)^{\eta_2} < 2^{\lambda^{k+1}} < (\log n)^{4c_k}$$

Now we prove

(58) $$N_1(\lambda_1, \cdots, \lambda_{k+1}) \leqq 2^{\sum_{i=1}^{k+1} \lambda_i} \left(\prod_{i=1}^{k} \lambda_i \right)^{-1} 2^{-\lambda_{k+1} 2^{k+2}} \ .$$

As in the proof of (48) we obtain that if (58) would not hold then there would be primes $p_i^{(\lambda)}$, $1 \leqq i \leqq k$, $\lambda = 1$ or 2 and two integers $t^{(1)}$ and $t^{(2)}$ so that the 2^{k+1} integers

$$t^{\lambda} \prod_{i=1}^{k} p_i^{(\lambda)}, \quad \lambda = 1 \text{ or } 2$$

all would be b's of the fifth class, but as we have already seen this implies $g(m) \geqq 2^k$ (for $m = t^{(1)}t^{(2)}\prod_{i=1}^{k} p_i^{(1)}p_i^{(2)}$). Thus (58) is proved. Now we obtain from (58) by a simple computation the details of which we supress that (the dash indicates that (57) is satisfied)

$$(59) \qquad \sum{}' N_1(\lambda_1, \cdots, \lambda_{k+1}) \leqq \quad 2^{\sum_{i=1}^{k+1} \lambda_i} \left(\prod_{i=1}^{k+1} \lambda_i \right)^{-1} 2^{-\lambda_{k+1}/2_{k+2}} = o\left(\frac{n}{\log n} \right).$$

(59) contradicts (56) and this contradiction proves (34) and also (33) and hence completes the proof of Theorem 3.

Let $2^{k-1} < l \leqq 2^k$. Theorem 3 could be sharpened to

$$u_l(n) = \frac{n(\log \log n)^{k-1}}{(k-1)! \log n} + 0\left(\frac{n}{(\log n)^{1+c}} \right)$$

where $c > 0$ is a suitable positive constant. But at present I can not prove for $l > 2$ a result as sharp as (4) and (5).

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